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M-theory and characteristic classes*

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Abstract: In this note we show that the Chern-Simons and the one-loop terms in the M-theory action can be written in terms of new characters involving the M-theory four-form and the string classes. This sheds a new light on the topological structure behind M-theory and suggests the construction of a theory of 'higher' characteristic classes.

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1. Introduction

M-theory has emerged during the last decade as the theory that unifies string theories and thus is the best candidate for a theory of quantum gravity [1 − 3]. This theory is believed to be very rich both physically and mathematically. However, such structures are only being discovered and a complete description (let alone an understanding) of the theory remains a grand challenge. The study of the partition function related to the four-form has uncovered deep connections to K-theory [4], and to a less-understood extent twisted K-theory [5] and elliptic cohomology [6].

In this note, we make further observations on the topological part of the action, that we hope will provide an approach that makes the topological structure more transparent. Our arguements are rather intuitive but seem to point to some deep mathematical theory. In particular, the structure of the action motivated us to propose defining new "higher" characteristic classes based on the Pontrjagin classes. We define the total string class and the string character and we encode the action in terms of a character built out of the four-form and the string classes.

The M-theory action contains the usual eleven-dimensional supergravity terms [7], namely the Einstein-Hilbert, the G_4 kinetic terms, the Rarita-Schwinger term, as well as the subtle topological terms, namely the Chern-Simons term and the one-loop term [8] given by (see [9] for a review)

$$\begin{aligned} S_{11} &= S_{CS} + S_{1-loop} \\ &= \int_{Y^{11}} \frac{1}{6} C_3 \wedge G_4 \wedge G_4 - \int_{Y^{11}} C_3 \wedge I_8(g) \end{aligned} \tag{1.1}$$

where $[I_8(g)] = \frac{p_2 - (p_1/2)^2}{48}$, written in terms of the Pontrjagin classes of the tangent bundle.

Using cobordism, Witten has uncovered a structure related to E_8 index theory by writing the above action on a twelve-dimensional manifold Z^{12} whose boundary is the M-theory eleven-manifold Y^{11}. In twelve dimensions, the topological part of the action is then

$$\begin{aligned} S_{12} &= \int_Z I_{12} \\ &= \int_Z \frac{1}{6} G_4 \wedge G_4 \wedge G_4 - G_4 \wedge I_8 \,. \end{aligned} \tag{1.2}$$

Of particular interest is the mod 6 congruence which was derived in [10] using physical arguments related to Hořava-Witten boundaries and in [11] using E_8 index theory.

2. The proposal

Here we propose writing the chern-Simons term in an exponentiated form as

$$\left[e^{G_4}\right]_{(12)} = \frac{1}{6} G_4 \wedge G_4 \wedge G_4 \,, \tag{2.1}$$

which looks like a character. Obviously, this very simply encodes the correct normalization factor.

Next we look at the one-loop term. Define the total string class in terms of the individual string classes as

$$\lambda = \lambda_0 + \lambda_1 + \lambda_2 + \cdots \tag{2.2}$$

where $\lambda_0 = 1$, $\lambda_1 = p_1/2$ is the usual string class, and we define λ_2 to be $p_2/2$, as the second string class, and so on. Define the exponentiated class (i.e. the "total string character"), e^{λ}, whose degree eight component gives

$$\frac{1}{2} \left(\lambda_1^2 - 2\lambda_2\right) . \tag{2.3}$$

The one-loop gravitational polynomial I_8, rewritten in terms of the introduced string classes λ_1 and λ_2, is then exactly minus 24 times the eight-form (2.3). The minus sign will be useful since I_8 shows up as in the action with the relative minus sign. The point out of this manipulation is that this is analogous to writing the second Chern character in terms of the Chern classes as

$$ch_2 = \frac{1}{2}(c_1^2 - 2c_2) \,. \tag{2.4}$$

Going back to the total topological action, this can be written as

$$\left[e^{G_4}\left[1 + \frac{1}{24}\left(e^{\lambda} - 1\right)\right]\right]_{(12)} , \tag{2.5}$$

the degree twelve component. This apparently predicts a term

$$\frac{1}{2} G_4 \wedge G_4 \wedge \frac{\lambda_1}{24} \,, \tag{2.6}$$

that can be written in terms of $\sqrt{\widehat{A}}$ (without overall factors),[1]

$$\left[\sqrt{\widehat{A}}\,\right]_{(4)} = -\frac{1}{48}p_1 = -\frac{1}{24}\left[e^{\lambda}\right]_{(4)} = \frac{1}{24}\left[e^{-\lambda}\right]_{(4)}. \tag{2.7}$$

We can find a rationale to exclude such terms (as well as terms containing λ_3) on the basis of parity. We want to retain the terms such that

$$e^{G_4}e^{\lambda} = -e^{-G_4}e^{\lambda} \tag{2.8}$$

which kills the terms with even number of G_4's and keeps the ones with odd number of G_4's. So the total topological action is given by the parity-odd part of the twelve-form component of the character.[2] Such parity conditions are in fact not so foreign since they show up in M-theory in the calculation of higher order corrections, including the one-loop term. However, mathematically, it does not seem that a priori we need to impose such conditions on (2.1), and later on (3.3), in order to talk about the cohomology theory below.

The phase of the M-theory partition function [4, 16, 17] would then be written as

$$\Phi(C_3) = (-1)^{\frac{1}{2}I_{R.S.}} \exp 2\pi i \left[\int_{Z^{12}} e^{G_4}\left[1 + \frac{1}{24}(e^{\lambda} - 1)\right]\right], \tag{2.9}$$

where it is understood that we pick the degree twelve component of the integrand. Note the sign ambiguity in the phase is canceled by the one coming from the phase of the Pfaffian of the Rarita-Schwinger operator given in terms of the Rarita-Schwinger action $I_{R.S.}$.

3. A cohomology theory?

Here we draw an analogy to the construction of the Chern character based on ordinary curvatures of connection one-forms. We will see that this suggests that an analogous theory based on the Pontrjagin classes (more precisely the string classes) instead of the Chern classes seems to emerge.

Recall the Chern character

$$ch(F_2) = e^{F_2} = rk + F_2 + \frac{1}{2}F_2 \wedge F_2 + \frac{1}{6}F_2 \wedge F_2 \wedge F_2 + \cdots \tag{3.1}$$

[1]In fact one can ask, in a different but related context, whether one can replace the K-theoretic formula $F(x) = ch(x)\sqrt{\widehat{A}(X)}$ for the RR fields [12] [13] by something like $F(x) = -\frac{1}{24}ch(x)e^{\lambda}$ (note the sign). They match for the lower two degrees of the gravitational part (i.e. zero and 4), but not the degree 8. We find this interesting but will not elaborate here on whether this is merely a coincidence in low degrees or whether it leads to a correction of the K-theoretic formula. The latter would be related to refinements proposed in [14]. We will explore this elsewhere.

[2]Note that Lagrangian of eleven-dimensional supergravity has a symmetry given by reversing the sign of the 3-form potential C_3, accompanied by a reversal in sign of odd number of space coordinates [15].

Now, in light of the above construction, we would like to do the same for G_4[3]

$$e^{G_4} = c + G_4 + \frac{1}{2} G_4 \wedge G_4 + \frac{1}{6} G_4 \wedge G_4 \wedge G_4 + \cdots \tag{3.2}$$

which we would like to think of as a sort of M-theoretic character $\mathbb{M}$, analogous to the Chern character. The term "c" should refer to the appropriate concept in this case that replaces the rank of the bundle for the case of K-theory.[4]

The E_8 and Rarita-Schwinger indices are then encoded in[5]

$$\text{Index (M object)} = \int \mathbb{M} = \int e^{G_4} . \tag{3.3}$$

The individual components are

$$\begin{aligned} \mathbb{M}_0 &= 1 \\ \mathbb{M}_1 &= G_4 \\ \mathbb{M}_2 &= \frac{1}{2} G_4 \wedge G_4 \\ \mathbb{M}_3 &= \frac{1}{6} G_4 \wedge G_4 \wedge G_4 , \end{aligned} \tag{3.4}$$

the latter three corresponding to the M2-brane, the M5-brane/little M-theory, and M-theory respectively (with the correct normalization!). We see that as a byproduct the M-branes and "M-theories" receive a somewhat unified treatment.

Let us explain this. The Chern-Simons construction for type IIA string theory and for the M-fivebrane go in a very similar way. In both we have a manifold of dimension $4k + 2$ with $k = 1$ for the fivebrane and $k = 2$ for type IIA string theory. The Chern-Simons construction requires extending the $(4k + 2)$-dimensional manifold X to a $(4k + 3)$-dimensional manifold $X \times S^1$ by an extra S^1, which, in the case of type IIA is just the extension to M-theory. Then the construction requires extending the resulting manifold to a $(4k + 4)$-dimensional "coboundary", i.e. an N such that $\partial N = X \times S^1$. In the case of IIA/M-theory, this is just the manifold Z we encountered in the introduction. In both cases one is extending the manifold together with a four-class, so that this requires the vanishing of the spin cobordism cohomology group $M\text{Spin}_{4k+3}(K(\mathbb{Z}, 4))$. This was shown

[3]Clearly, the Chern character should be written in terms of Lie-algebra valued curvature, whose trace we suppress in an obvious way. The character corresponding to G_4 does not seem to have such a valuedness since G_4 itself does not correspond to structure group in a literal way. Of course, it is true that if it is to be a (2-)gerbe then the simplest way is to take it to be abelian and thus corresponding to a U(1) in an appropriate sense. If one writes the C-field in terms of E_8 gauge fields (e.g. take $C_3 = CS_3(A)$ literally as in [18] or in a more refined way as in [16]) then G_4 might have some nonabelian aspect to it. In any case, we leave such possibilities open.

[4]We do not see any reason for the constant term to be different from one, otherwise the normalizations are altered. However, the formalism can accomodate other possibilities.

[5]In the definition of $\mathbb{M}$ we have to also insert the gravitational correction term containing e^{λ} to be able to get the total action and thus for the statement to be correct. We however look only at G_4 at this stage and adding the gravitational term is straightforward. We simply replace $\mathbb{M}$ by $\mathbb{M}' = \mathbb{M}\left[1 + \frac{1}{24}\left(e^{\lambda} - 1\right)\right]$. Again, exclusion of some terms can be based on parity: odd (even) parity for odd- (even-)rank character.

to be the case by Stong for $k = 2$ [19] and by Hopkins-Singer for $k = 1$ [20]. Now, similarly to the case of the Chern-Simons term in M-theory, we propose writing the corresponding quadratic term involving only G_4 on N_8 in a similar exponential way, i.e

$$\left[e^{G_4}\right]_{(8)} = \frac{1}{2} G_4 \wedge G_4 \tag{3.5}$$

which is the formula derived by Witten [21] and also in a more general situation by Hopkins-Singer [20].[6][7] Another way of looking at the fivebrane in eleven dimensions is using tubular neighborhoods and disk bundles as in [22].

The above suggests that there is a generalized cohomology theory in which G_4 lives and the character is a multiplicative map from the theory of M-objects to $4k$-th cohomology.

$$\mathbb{M} = e^{G_4} : \mathcal{M} \to H^{4k} \tag{3.6}$$

where $\mathcal{M}$ is the (generalized cohomology?) theory that describes M-theory.

Recall that the Chern character is a map from K-theory to even cohomology

$$ch : K \to H^{ev} \tag{3.7}$$

and satisfies (for two vector bundles E and F)

$$ch(E \oplus F) = ch(E) \oplus ch(F) \tag{3.8}$$
$$ch(E \otimes F) = ch(E) \wedge ch(F)\,. \tag{3.9}$$

For $\mathcal{E}$ and $\mathcal{F}$ "M-objects", we want $\mathbb{M}$ to have properties analogous to those of the Chern character, i.e.

$$\begin{aligned} \mathbb{M}(\mathcal{E} \oplus \mathcal{F}) &= \mathbb{M}(\mathcal{E}) \oplus \mathbb{M}(\mathcal{F}) \\ \mathbb{M}(\mathcal{E} \otimes \mathcal{F}) &= \mathbb{M}(\mathcal{E}) \wedge \mathbb{M}(\mathcal{F})\,. \end{aligned} \tag{3.10}$$

Note further that $d\mathbb{M} = 0$ since G_4 itself is closed by virtue of the Bianchi identity.

We have proposed a theory of multiplicative classes based on the Pontrjagin classes (more precisely, the string classes, the higher ranks of which we defined above) instead of the usual Chern characters which are built out of the Chern classes. For K-theory, the objects corresponding to the two-form curvature F_2 are vector bundles. What are the corresponding M-objects, i.e. the ones related to G_4? We do not have a precise answer but we propose that they would have to do with '2-objects', e.g. 2-gerbes or more precise refinements as in [16]. This will be discussed seperately.

What is the theory we are looking for? From the general structure of the character, from the mod 24 congruence of the string character, and from the previous work [6, 23, 14], we expect such a theory to be some form/refinement of elliptic cohomology (e.g. related to the theory of topological modular forms). From the mathematical point of view, in order to have a theory of *characteristic* classes, we need to have a classifying space from which we pullback structures to our spaces. This, and the relation to the E_8 and Rarita-Schwinger bundles, will also be explored seperately.

[6]The two references differ by an overall minus sign, as pointed out in [20].

[7]There is another term in the fivebrane action extended to 8-dimensions. This is $G_4 \wedge \lambda_1$. To include this term it seem that one has to look at the combination $e^{G_4} e^{\lambda}$ in this case.

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